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The forcing vertex detour monophonic number of a graph ☆

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Abstract

For any two vertices x and y in a connected graph G, an x–y path is a monophonic path if it contains no chord, and a longest x–y monophonic path is called an x–y detour monophonic path. For any vertex x in G, a set $S_x \subseteq V(G)$ is an x-detour monophonic set of G if each vertex $v \in V(G)$ lies on an x–y detour monophonic path for some element y in S_x. The minimum cardinality of an x-detour monophonic set of G is the x-detour monophonic number of G, denoted by $dm_x(G)$. A subset T_x of a minimum x-detour monophonic set S_x of G is an x-forcing subset for S_x if S_x is the unique minimum x-detour monophonic set containing T_x. An x-forcing subset for S_x of minimum cardinality is a minimum x-forcing subset of S_x. The forcing x-detour monophonic number of S_x, denoted by $f_{dm_x}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The forcing x-detour number of G is $f_{dm_x}(G) = \min\{f_{dm_x}(S_x)\}$, where the minimum is taken over all minimum x-detour monophonic sets S_x in G. We determine bounds for it and find the same for some special classes of graphs. Also we show that for every pair s, t of integers with $2 \leq s \leq t$, there exists a connected graph G such that $f_{dm_x}(G) = s$ and $dm_x(G) = t$ for some vertex x in G.

Keywords: Detour monophonic path; Vertex detour monophonic number; Forcing vertex detour monophonic number

1. Introduction

By a *graph* $G = (V, E)$ we mean a non-trivial finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to Harary [1]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic

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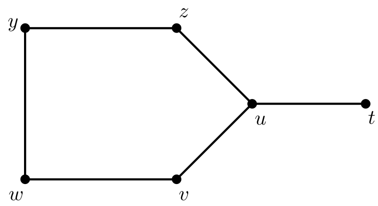

Fig. 2.1. The graph G in Example 2.2.

set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-*set* of G. The geodetic number of a graph was introduced in [2] and further studied in [3–5].

The concept of vertex geodomination in graphs was introduced in [6] and further studied in [7]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-*geodominating set* of G if each vertex v of G lies on an $x-y$ geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the x-*geodomination number* of G and is denoted by $g_x(G)$.

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x-y$ monophonic path P is called an $x-y$ *detour monophonic path*. The *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophonic path (or $u-v$ detour monophonic path) in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m G$ of G is $rad_m G = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m G$ of G is $diam_m G = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [8] and further studied in [9].

The concept of vertex detour monophonic number was introduced in [10]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-*detour monophonic set* of G if each vertex v of G lies on an $x-y$ detour monophonic path in G for some element y in S. The minimum cardinality of an x-detour monophonic set of G is defined as the x-*detour monophonic number* of G and is denoted by $dm_x(G)$. An x-detour monophonic set of cardinality $dm_x(G)$ is called a dm_x-set of G.

2. Forcing vertex detour monophonic number

Let x be any vertex of a connected graph G. Although G contains a minimum x-detour monophonic set there are connected graphs which may contain more than one minimum x-detour monophonic set. For example the graph G given in Fig. 2.1 contains more than one minimum x-detour monophonic set. For each minimum x-detour monophonic set S_x in a connected graph G there is always some subset T_x of S_x that uniquely determines S_x as the minimum x-detour monophonic set containing T_x. Such sets are called "vertex forcing subsets" and we discuss these sets in this section. Also, forcing concepts have been studied for such diverse parameters in graphs as the graph reconstruction number [11], the domination number [12], and the geodetic number [13].

Definition 2.1. Let x be a vertex of a connected graph G and let S_x be a minimum x-detour monophonic set of G. A subset T_x of S_x is called an x-*forcing subset* for S_x if S_x is the unique minimum x-detour monophonic set containing T_x. An x-forcing subset for S_x of minimum cardinality is a *minimum x-forcing subset* of S_x. The *forcing x-detour monophonic number* of S_x, denoted by $f_{dm_x}(S_x)$, is the cardinality of a minimum x-forcing subset of S_x. The *forcing x-detour monophonic number* of G is $f_{dm_x}(G) = min\{f_{dm_x}(S_x)\}$, where the minimum is taken over all minimum x-detour monophonic sets S_x in G.

Example 2.2. For the graph G given in Fig. 2.1, the minimum vertex detour monophonic sets, the vertex detour monophonic numbers, the minimum forcing vertex detour monophonic sets and the forcing vertex detour monophonic numbers are given in Table 2.1.

The next theorem immediately follows from the definition of x-detour monophonic number and forcing x-detour monophonic number of a graph G.

Theorem 2.3. *For any vertex x in a connected graph G, $0 \leq f_{dm_x}(G) \leq dm_x(G)$.*

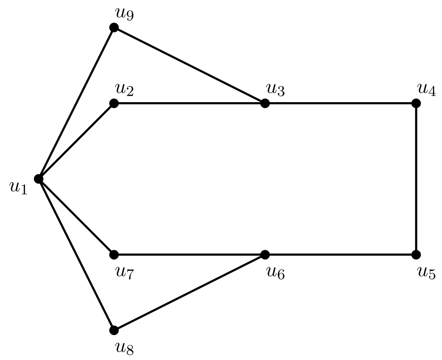

Fig. 2.2. A graph G with $f_{dm_{u_1}}(G) = dm_{u_1}(G) = 2$.

Table 2.1
The forcing vertex detour monophonic numbers of the graph G given in Fig. 2.1.

Vertex x	Minimum x-detour monophonic sets	$dm_x(G)$	Minimum forcing x-detour monophonic sets	$f_{dm_x}(G)$
t	$\{v, w\}, \{y, z\}, \{w, y\}$	2	$\{v\}, \{z\}$	1
u	$\{t, v, w\}, \{t, y, z\}, \{t, w, y\}$	3	$\{v\}, \{z\}$	1
v	$\{t, z\}$	2	$\emptyset$	0
w	$\{t, v\}, \{t, z\}$	2	$\{v\}, \{z\}$	1
y	$\{t, v\}, \{t, z\}$	2	$\{v\}, \{z\}$	1
z	$\{t, v\}$	2	$\emptyset$	0

The bounds in Theorem 2.3 are sharp. For the graph G given in Fig. 2.1, $f_{dm_v}(G) = 0$. Also, for the graph G given in Fig. 2.2, $S_1 = \{u_3, u_5\}$, $S_2 = \{u_3, u_6\}$, $S_3 = \{u_4, u_5\}$ and $S_4 = \{u_4, u_6\}$ are the minimum u_1-detour monophonic sets of G and so $dm_{u_1}(G) = 2$. It is easily verified that no minimum u_1-detour monophonic set is the unique minimum u_1-detour monophonic set containing any of its proper subsets. It follows that $f_{dm_{u_1}}(G) = 2$ and hence $dm_{u_1}(G) = f_{dm_{u_1}}(G)$. The inequalities in Theorem 2.3 can be strict. For the graph G given in Fig. 2.1, $dm_u(G) = 3$ and $f_{dm_u}(G) = 1$. Thus $0 < f_{dm_u}(G) < dm_u(G)$.

In the following theorem we characterize graphs G for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{dm_x}(G) = 1$.

Theorem 2.4. *Let x be a vertex of a connected graph G. Then*

(i) $f_{dm_x}(G) = 0$ *if and only if G has a unique minimum x-detour monophonic set.*

(ii) $f_{dm_x}(G) = 1$ *if and only if G has at least two minimum x-detour monophonic sets and one of which is a unique minimum x-detour monophonic set containing one of its elements.*

(iii) $f_{dm_x}(G) = dm_x(G)$ *if and only if no minimum x-detour monophonic set of G is the unique minimum x-detour monophonic set containing any of its proper subsets.*

Definition 2.5. A vertex u of a connected graph G is said to be an *x-detour monophonic vertex* of G if u belongs to every minimum x-detour monophonic set of G.

For the graph G in Fig. 2.3, $S_1 = \{s, z, w\}$ and $S_2 = \{s, z, v\}$ are the minimum t-detour monophonic sets and so s and z are the t-detour monophonic vertices of G. In particular, every extreme vertex of G other than x is an x-detour monophonic vertex of G.

The following theorem and corollary follows immediately from the definitions of an x-detour monophonic vertex and forcing x-detour monophonic subset of G.

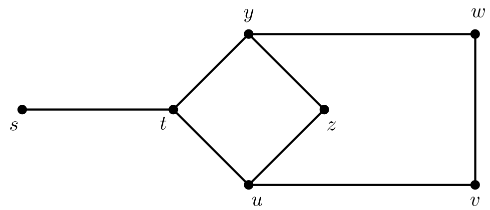

Fig. 2.3. A graph G with vertex detour monophonic vertices.

Theorem 2.6. *Let x be a vertex of a connected graph G and let $\mathcal{F}_{dm_x}$ be the set of relative complements of the minimum x-forcing subsets in their respective minimum x-detour monophonic sets in G. Then $\bigcap_{F\in\mathcal{F}_{dm_x}} F$ is the set of x-detour monophonic vertices of G.*

Corollary 2.7. *Let G be a connected graph and S_x a minimum x-detour monophonic set of G. Then no x-detour monophonic vertex of G belongs to any minimum x-forcing subset of S_x.*

Theorem 2.8. *Let x be a vertex of a connected graph G and let M_x be the set of all x-detour monophonic vertices of G. Then $0 \leq f_{dm_x}(G) \leq dm_x(G) - |M_x|$.*

Proof. Let S_x be any minimum x-detour monophonic set of G. Then $dm_x(G) = |S_x|$, $M_x \subseteq S_x$ and S_x is the unique minimum x-detour monophonic set containing $S_x - M_x$ and so $f_{dm_x}(G) \leq |S_x - M_x| = dm_x(G) - |M_x|$. □

Corollary 2.9. *If G is a complete graph or a tree or a complete bipartite graph, then $f_{dm_x}(G) = 0$ for any vertex x in G.*

Proof. It is easily seen that G has unique minimum x-detour monophonic set for any vertex x in G. Hence the result follows from Theorem 2.4(i). □

Theorem 2.10. *For any vertex x in the cycle C_n of order $n \geq 4$, $fdm_x(C_n) = 0$ or 1 according as n is even or odd.*

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order $n \geq 4$. Let x be any vertex in C_n, say $x = u_1$. If n is even, then $S_x = \{u_{\frac{n}{2}+1}\}$ is the unique minimum x-detour monophonic set of C_n and so by Theorem 2.4(i), $f_{dm_x}(C_n) = 0$. If n is odd, then $S_{x_1} = \{u_2, u_3\}$, $S_{x_2} = \{u_n, u_{n-1}\}$ and $S_{x_3} = \{u_i, u_j : 3 \leq i \leq \frac{n+1}{2}$ and $\frac{n+3}{2} \leq j \leq n-1\}$ are the minimum x-detour monophonic sets of C_n. Hence it follows from Theorem 2.4(i) that $f_{dm_x}(C_n) \neq 0$. Clearly, $T_{x_1} = \{u_2\}$ is a minimum x-forcing subset of S_{x_1} and so $f_{dm_x}(C_n) = 1$. □

Theorem 2.11. *Let $W_n = K_1 + C_{n-1} (n \geq 5)$ be the wheel.*
(i) *If n is odd, then $f_{dm_x}(W_n) = 0$ for any vertex x in W_n.*
(ii) *If n is even, then $f_{dm_x}(W_n) = 0$ or 1 according as x is K_1 or x is in C_{n-1}.*

Proof. Let $C_{n-1} : u_1, u_2, \ldots, u_{n-1}, u_1$ be a cycle of order $n - 1$ and u the vertex of K_1. If n is odd, then W_n has the unique minimum x-detour monophonic set of W_n for any vertex x in W_n and so by Theorem 2.4(i), $f_{dm_x}(W_n) = 0$. Now, assume that n is even. If $x = u$, then $S = \{u_1, u_2, \ldots, u_{n-1}\}$ is the unique minimum x-detour monophonic set of W_n and so by Theorem 2.4(i), $f_{dm_x}(W_n) = 0$. If $x \in V(C_{n-1})$, say $x = u_1$, then $S_{x_1} = \{u, u_2, u_3\}$, $S_{x_2} = \{u, u_{n-1}, u_{n-2}\}$ and $S_{x_3} = \{u, u_i, u_j : 3 \leq i \leq \frac{n}{2}$ and $\frac{n+2}{2} \leq j \leq n-2\}$ are the minimum x-detour monophonic sets of W_n. Hence it follows from Theorem 2.4(i) that $f_{dm_x}(W_n) \neq 0$. Clearly, $T_{x_1} = \{u_2\}$ is a minimum x-forcing subset of S_{x_1} and so $f_{dm_x}(W_n) = 1$. □

We proved in Theorem 2.3 that $0 \leq f_{dm_x}(G) \leq dm_x(G)$ for any vertex x in G. Already we have seen that if G is a complete graph K_{t+1}, then $f_{dm_x}(G) = 0$ and $dm_x(G) = t$ for any vertex x in G. Also for the graph G given in Fig. 2.4, $f_{dm_x}(G) = 1$ and $dm_x(G) = t \geq 2$ for the vertex x. The following theorem gives a realization for these parameters when $2 \leq s \leq t$.

Theorem 2.12. *For every pair s, t of integers with $2 \leq s \leq t$, there exists a connected graph G such that $f_{dm_x}(G) = s$ and $dm_x(G) = t$ for some vertex x in G.*

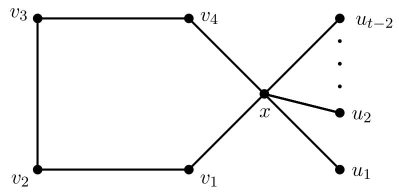

Fig. 2.4. A graph G with $f_{dm_x}(G) = 1$ and $dm_x(G) = t$.

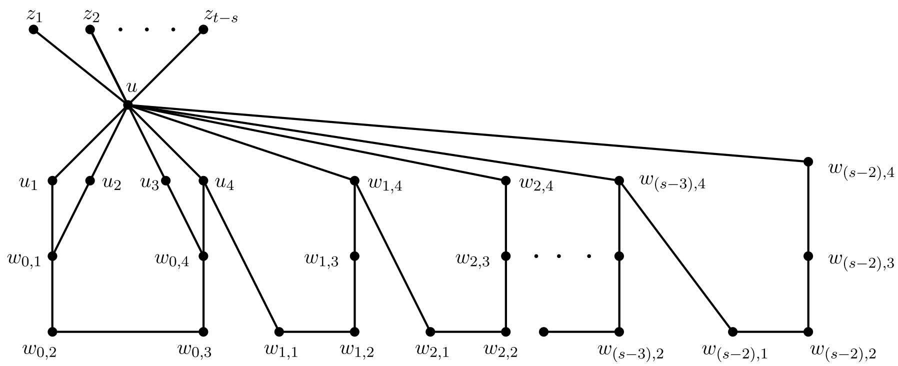

Fig. 2.5. A graph G with $f_{dm_x}(G) = s$ and $dm_x(G) = t$.

Proof. For each integer i with $0 \leq i \leq s-2$, let $F_i : w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}$ be a path of order 4 and let $K_{1,4}$ be a star with $V(K_{1,4}) = \{u, u_1, u_2, u_3, u_4\}$. Let H be the graph obtained from F_i and $K_{1,4}$ by (i) join $w_{0,1}$ with u_1 and u_2 in $K_{1,4}$ (ii) join $w_{0,4}$ with u_3 and u_4 in $K_{1,4}$ (iii) join each $w_{i,4}$ $(1 \leq i \leq s-2)$ with u in $K_{1,4}$ (iv) join each $w_{i,1}$ $(2 \leq i \leq s-2)$ with $w_{i-1,4}$ and (v) join $w_{1,1}$ with u_4 in $K_{1,4}$. Now, let G be the graph obtained from H by adding $t-s$ new vertices $z_1, z_2, \ldots, z_{t-s}$ and joining each $z_i (1 \leq i \leq t-s)$ with u. The graph G is shown in Fig. 2.5.

Let $x = u$ and let $S = \{z_1, z_2, \ldots, z_{t-s}\}$ be the set of all extreme vertices of G. Let $S_i = \{w_{i,1}, w_{i,2}\}$, $(1 \leq i \leq s-2)$, $S_{s-1} = \{w_{0,1}, w_{0,2}\}$ and $S_s = \{w_{0,3}, w_{0,4}\}$. First we claim that $dm_x(G) = t$. Now, we observe that a set S_x of vertices of G is a dm_x-set if and only if S_x contains exactly one vertex from each set $S_i (1 \leq i \leq s)$ and S_x contains S so that $dm_x(G) \geq t$. Since $S'_x = S \cup \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \ldots, w_{(s-3),1}, w_{(s-2),1}\}$ is an x-detour monophonic set of G with $|S'_x| = t$, it follows that $dm_x(G) = t$.

Next, we prove that $f_{dm_x}(G) = s$. Let $S_x = S \cup \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \ldots, w_{(s-3),1}, w_{(s-2),1}\}$ be a dm_x-set of G and let T_x be any minimum x-forcing subset of S_x. Then, by Theorem 2.8, we get $|T_x| \leq t - (t-s) = s$.

If $|T_x| < s$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_i$ for some $i = 1, 2, 3, \ldots, s$, say $y = w_{1,1}$. Let $S^1_x = (S_x - \{w_{1,1}\}) \cup \{w_{1,2}\}$. Then $S^1_x \neq S_x$ and S^1_x is also a minimum x-detour monophonic set of G such that it contains T_x, which is a contradiction to T_x an x-forcing subset of S_x. Thus $|T_x| = s$ and so $f_{dm_x}(G) = s$. $\square$

3. Upper forcing vertex detour monophonic number

The cardinality of vertex forcing subsets depend on the corresponding minimum vertex detour monophonic set of G. In this observation, we discuss these sets with minimum cardinality in the previous section. Now, we deal about vertex forcing subsets with maximum cardinality in this section.

Definition 3.1. Let x be any vertex of a connected graph G. The *upper forcing x-detour monophonic number*, $f^+_{dm_x}(G)$, of G is the maximum forcing x-detour monophonic number among all minimum x-detour monophonic sets of G.

Example 3.2. For the graph G given in Fig. 2.1, the minimum vertex detour monophonic sets, the vertex detour monophonic numbers, the forcing vertex detour monophonic sets, the minimum forcing vertex detour monophonic numbers and upper forcing vertex detour monophonic numbers are given in Table 3.1.

Table 3.1
The upper forcing vertex detour monophonic numbers of the graph G given in Fig. 2.1.

Vertex x	dm_x-sets	$dm_x(G)$	x-forcing subsets	$f_{dm_x}(G)$	$f^+_{dm_x}(G)$
t	$\{v, w\}, \{y, z\}, \{w, y\}$	2	$\{v\}, \{z\}, \{w, y\}$	1	2
u	$\{t, v, w\}, \{t, y, z\}, \{t, w, y\}$	3	$\{v\}, \{z\}, \{w, y\}$	1	2
v	$\{t, z\}$	2	$\emptyset$	0	0
w	$\{t, v\}, \{t, z\}$	2	$\{v\}, \{z\}$	1	1
y	$\{t, v\}, \{t, z\}$	2	$\{v\}, \{z\}$	1	1
z	$\{t, v\}$	2	$\emptyset$	0	0

From Table 3.1, the forcing vertex detour monophonic number and the upper forcing vertex detour monophonic number of a graph are different.

Next we present two theorems whose routine proof is omitted.

Theorem 3.3. *For any vertex x in a connected graph G, $0 \le f_{dm_x}(G) \le f^+_{dm_x}(G) \le dm_x(G)$.*

Theorem 3.4. *Let x be a vertex of a connected graph G. Then*
(i) *$f^+_{dm_x}(G) = 0$ if and only if G has a unique minimum x-detour monophonic set.*
(ii) *$f^+_{dm_x}(G) = 1$ if and only if G has at least two minimum x-detour monophonic sets and every dm_x-set is the unique dm_x-set containing one of its elements.*
(iii) *$f^+_{dm_x}(G) = dm_x(G)$ if and only if no minimum x-detour monophonic set of G is the unique minimum x-detour monophonic set containing any of its proper subsets.*

Theorem 3.5. *For any vertex x in the cycle C_n of order $n \ge 4$, $fdm^+_x(C_n) = 0$ or 2 according as n is even or odd.*

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order $n \ge 4$. Let x be any vertex in C_n, say $x = u_1$. If n is even, then $S_x = \{u_{\frac{n}{2}+1}\}$ is the unique minimum x-detour monophonic set of C_n and so by Theorem 3.4(i), $f^+_{dm_x}(C_n) = 0$. If n is odd, then $S_{x_1} = \{u_2, u_3\}$, $S_{x_2} = \{u_n, u_{n-1}\}$ and $S_{x_3} = \{u_i, u_j : 3 \le i \le \frac{n+1}{2}$ and $\frac{n+3}{2} \le j \le n-1\}$ are the minimum x-detour monophonic sets of C_n. Hence it follows from Theorem 3.4(i) that $f^+_{dm_x}(C_n) \ne 0$. Also, $T_x = \{u_{\frac{n+1}{2}}, u_{\frac{n+3}{2}}\}$ is one of the minimum x-detour monophonic set of C_n. It is easily verified that any proper subset of T_x is contained in some other minimum x-detour monophonic set of G and so $f^+_{dm_x}(C_n) = |T_x| = 2$. □

Theorem 3.6. *Let $W_n = K_1 + C_{n-1} (n \ge 5)$ be the wheel.*
(i) *If n is odd, then $f^+_{dm_x}(W_n) = 0$ for any vertex x in W_n.*
(ii) *If n is even, then $f^+_{dm_x}(W_n) = 0$ or 2 according as x is K_1 or x is in C_{n-1}.*

Proof. Let $C_{n-1} : u_1, u_2, \ldots, u_{n-1}, u_1$ be a cycle of order $n-1$ and let u be the vertex of K_1. If n is odd, then W_n has the unique dm_x-set for any vertex x in W_n and so by Theorem 3.4(i), $f^+_{dm_x}(W_n) = 0$. Now, assume that n is even. If $x = u$, then dm_x-set is unique and so $f^+_{dm_x}(W_n) = 0$. If $x \in V(C_{n-1})$, say $x = u_1$, then $S_{x_1} = \{u, u_2, u_3\}$, $S_{x_2} = \{u, u_{n-2}, u_{n-1}\}$ and $S_{x_3} = \{u, u_i, u_j : 3 \le i \le \frac{n}{2}$ and $\frac{n+2}{2} \le j \le n-2\}$ are the minimum x-detour monophonic sets of W_n. Hence it follows that $T_x = \{u_{\frac{n}{2}}, u_{\frac{n+2}{2}}\}$ is the unique forcing x-detour monophonic set of the minimum x-detour monophonic set $S = \{u, u_{\frac{n}{2}}, u_{\frac{n+2}{2}}\}$ so that $f^+_{dm_x}(W_n) \ge |T_x| = 2$. Since u is the x-detour monophonic vertex of G, by Corollary 2.7, we have $f^+_{dm_x}(G) = 2$. □

Note that we have $0 \le f_{dm_x}(G) \le f^+_{dm_x}(G) \le dm_x(G)$ for every connected graph G. The following theorem gives a realization result for the parameters $f_{dm_x}(G)$, $f^+_{dm_x}(G)$ and $dm_x(G)$.

Theorem 3.7. *For any three positive integers r, s and t with $2 \le r \le s \le t$ and $2r - s \ge 0$, there exists a connected graph G with $f_{dm_x}(G) = r$, $f^+_{dm_x}(G) = s$ and $dm_x(G) = t$ for some vertex x in G.*

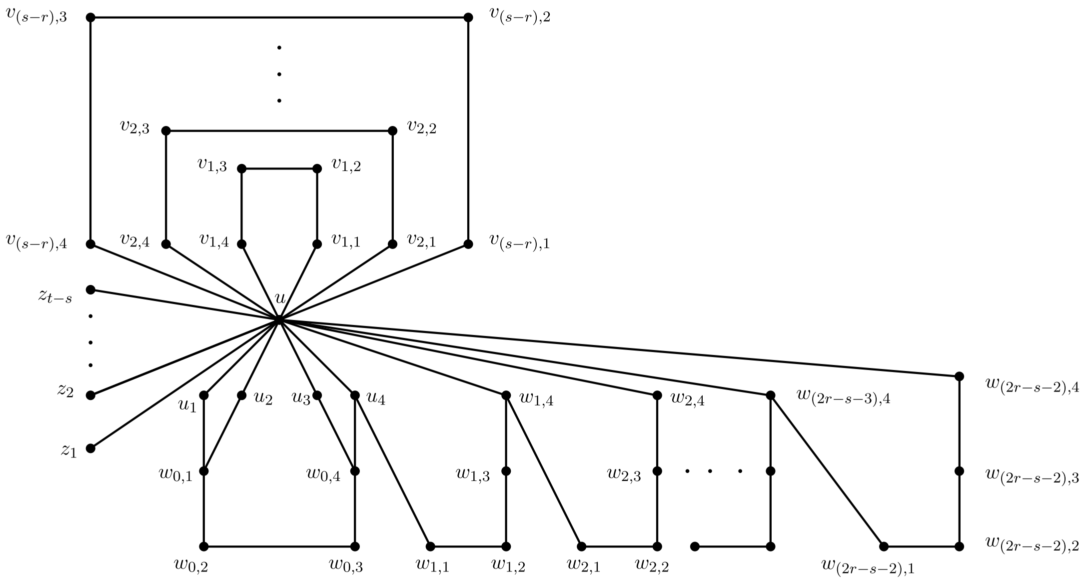

Fig. 3.1. The graph G in Case 1 of Theorem 3.7 with $f_{dm_x}(G) = r$, $f^+_{dm_x}(G) = s$ and $dm_x(G) = t$.

Proof. We prove this theorem by considering 3 cases.

Case 1. $2 \le r \le s \le 2r - 2$. For each integer i with $0 \le i \le 2r - s - 2$, let $F_i : w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}$ be a path of order 4, for each integer j with $1 \le j \le s - r$, let $H_j : v_{j,1}, v_{j,2}, v_{j,3}, v_{j,4}$ be another path of order 4 and let $K_{1,4}$ be a star with $V(K_{1,4}) = \{u, u_1, u_2, u_3, u_4\}$. Let H be the graph obtained from F_i, H_j and $K_{1,4}$ by (i) join $w_{0,1}$ with u_1 and u_2 in $K_{1,4}$ (ii) join $w_{0,4}$ with u_3 and u_4 in $K_{1,4}$ (iii) join each $w_{i,4} (1 \le i \le 2r - s - 2)$ with u in $K_{1,4}$ (iv) join each $w_{i,1} (2 \le i \le 2r - s - 2)$ with $w_{i-1,4}$ (v) join $w_{1,1}$ with u_4 in $K_{1,4}$ and (vi) join each $v_{j,i}, v_{j,4} (1 \le j \le s - r)$ with u in $K_{1,4}$. Now, let G be the graph obtained from H by adding $t - s$ new vertices $z_1, z_2, \dots, z_{t-s}$ and joining each $z_i (1 \le i \le t - s)$ with u. The graph G is shown in Fig. 3.1.

Let $x = u$ and let $S = \{z_1, z_2, \dots, z_{t-s}\}$ be the set of all extreme vertices of G. Let $S_i = \{w_{i,1}, w_{i,2}\} (1 \le i \le 2r - s - 2)$, $S_{2r-s-1} = \{w_{0,1}, w_{0,2}\}$ and $S_{2r-s} = \{w_{0,3}, w_{0,4}\}$. Also, for $1 \le i \le s - r$, let $S'_j = \{v_{j,1}, v_{j,2}\}$, $S''_j = \{v_{j,2}, v_{j,3}\}$ and $S'''_j = \{v_{j,3}, v_{j,4}\}$. Now, we observe that a set S_x of vertices of G is a dm_x-set if S_x contains exactly one vertex from each set $S_i (1 \le i \le 2r - s - 2)$, S_x contains S and S_x contains for each $j (1 \le j \le s - r)$ exactly one set from S'_j, S''_j, S'''_j. Hence $dm_x(G) \ge t$. Since $S'_x = S \cup S'_1 \cup S'_2 \cup \dots \cup S'_{(s-r)} \cup \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \dots, w_{(2r-s-2),1}\}$ is an x-detour monophonic set of G with $|S'_x| = t$, it follows that $dm_x(G) = t$.

Next, we prove that $f_{dm_x}(G) = r$ and $f^+_{dm_x}(G) = s$. Let T_x be any minimum x-forcing subset of S'_x. Then by Theorem 2.8, $|T_x| \le t - (t - s) = s$. Since any subset of a minimum x-detour monophonic set with cardinality less than r contained in more than one minimum x-detour monophonic set of G, we have $f_{dm_x}(G) \ge r$. It is easily seen that $T'_x = \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \dots, w_{(2r-s-2),1}, v_{1,1}, v_{2,1}, \dots, v_{(s-r),1}\}$ is a minimum x-forcing subset of a minimum x-detour monophonic set $S'_x = S \cup S'_1 \cup S'_2 \cup \dots \cup S'_{(s-r)} \cup \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \dots, w_{(2r-s-2),1}\}$ and $T''_x = \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \dots, w_{(2r-s-2),1}, v_{1,2}, v_{1,3}, v_{2,2}, v_{2,3}, \dots, v_{(s-r),2}, v_{(s-r),3}\}$ is a minimum x-forcing subset of a minimum x-detour monophonic set $S''_x = S \cup S''_1 \cup S''_2 \cup \dots \cup S''_{(s-r)} \cup \{w_{0,2}, w_{0,3}, w_{1,1}, w_{2,1}, \dots, w_{(2r-s-2),1}\}$ of G. It follows that $f_{dm_x}(G) = r$ and $f^+_{dm_x}(G) = s$.

Case 2. $s = 2r - 1$. For each integer i with $1 \le i \le 3$, let $F_i : u_{i,1}, u_{i,2}, u_{1,3}, u_{1,4}$ be a path of order 4. Let H be the graph obtained from $F_i (1 \le i \le 3)$ by (i) join $u_{2,1}$ with $u_{1,1}$ and $u_{1,4}$ (ii) join $u_{2,4}$ with $u_{1,4}, u_{3,2}$ and $u_{3,4}$ (iii) join $u_{3,1}$ with $u_{1,4}$ and (iv) join $u_{3,4}$ with $u_{2,2}$. For each integer i with $1 \le i \le r - 1$, let $H_i : w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}$ be a path of order 4. Let G' be the graph obtained from H and H_i by joining each $w_{i,1}$ and $w_{i,4}$ with $u_{1,1}$ in H. The graph G is obtained from G' by adding $t - s$ new vertices $z_1, z_2, \dots, z_{t-s}$ and joining each z_i with $u_{1,1}$ in G'. The graph G is shown in Fig. 3.2.

Let $x = u_{1,1}$ and let $S = \{z_1, z_2, \dots, z_{t-s}\}$ be the set of all extreme vertices of G. Also, for $1 \le i \le r - 1$, let $S'_i = \{v_{i,1}, v_{i,2}\}$, $S''_i = \{v_{i,2}, v_{i,3}\}$ and $S'''_i = \{v_{i,3}, v_{i,4}\}$. Now, we observe that a set S_x of vertices of G is a dm_x-set if S_x contains exactly one vertex from $\{u_{3,2}, u_{3,3}\}$ and S_x contains S, and for each $i (1 \le i \le r - 1)$ exactly one set

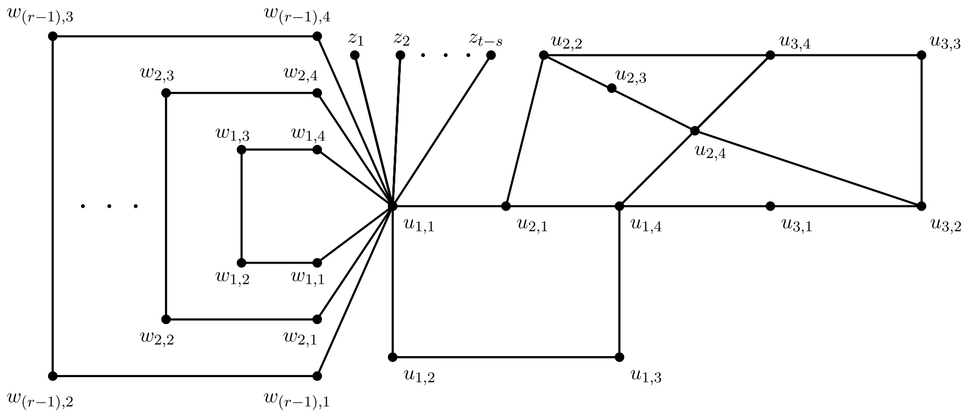

Fig. 3.2. The graph G in Case 2 of Theorem 3.7 with $f_{dm_x}(G) = r$, $f^+_{dm_x}(G) = s$ and $dm_x(G) = t$.

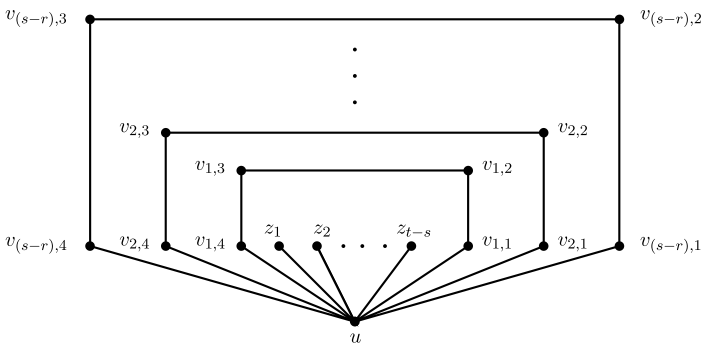

Fig. 3.3. The graph G in Case 3 of Theorem 3.7 with $f_{dm_x}(G) = r$, $f^+_{dm_x}(G) = s$ and $dm_x(G) = t$.

from S'_i, S''_i, S'''_i. Hence $dm_x(G) \geq t$. Since $S'_x = S \cup \{u_{3,3}\} \cup S'_1 \cup S'_2 \cup \cdots \cup S'_{(r-1)}$ is an x-detour monophonic set of G with $|S'_x| = t$, it follows that $dm_x(G) = t$.

Next, we prove that $f_{dm_x}(G) = r$ and $f^+_{dm_x}(G) = s$. Let T_x be any minimum x-forcing subset of S'_x. Then by Theorem 2.8, $|T_x| \leq t - (t - s) = s$. Since any subset of a minimum x-detour monophonic set with cardinality less than r contained in more than one minimum x-detour monophonic set of G, we have $f_{dm_x}(G) \geq r$. It is easily seen that $T'_x = \{u_{3,3}\} \cup \{w_{1,1}, w_{2,1}, \ldots, w_{(r-1),1}\}$ is a minimum x-forcing subset of a minimum x-detour monophonic set $S'_x = S \cup \{u_{3,3}\} \cup S'_1 \cup S'_2 \cup \cdots \cup S'_{(r-1)}$ and $T''_x = \{u_{3,3}\} \cup \{w_{1,2}, w_{1,3}, w_{2,2}, w_{2,3} \ldots, w_{(r-1),2}, w_{(r-1),3}\}$ is a minimum x-forcing subset of a minimum x-detour monophonic set $S''_x = S \cup \{u_{3,3}\} \cup S''_1 \cup S''_2 \cup \cdots \cup S''_{(r-1)}$ of G. It follows that $f_{dm_x}(G) = r$ and $f^+_{dm_x}(G) = 2r - 1 = s$.

Case 3. $s = 2r$. For each i with $1 \leq i \leq s - r$, let $F_i : v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ be a path of order 4 and let $K_{1,t-s}$ be a star with $V(K_{1,t-s}) = \{u, z_1, z_2, \ldots, z_{t-s}\}$. Let G be the graph obtained from F_i and $K_{1,t-s}$ by (i) join $v_{i,1}(1 \leq i \leq s-r)$ with u in $K_{1,t-s}$ and (ii) join $v_{i,4}(1 \leq i \leq s - r)$ with u in $K_{1,t-s}$. The graph G is shown in Fig. 3.3.

Let $x = u$ and let $S = \{z_1, z_2, \ldots, z_{t-s}\}$ be the set of all extreme vertices of G. For $1 \leq i \leq s - r$, let $S'_j = \{v_{j,1}, v_{j,2}\}$, $S''_j = \{v_{j,2}, v_{j,3}\}$ and $S'''_j = \{v_{j,3}, v_{j,4}\}$. Now, we observe that a set S_x of vertices of G is a dm_x-set if S_x contains S and S_x contains for each $j(1 \leq j \leq s - r)$ exactly one set from S'_j, S''_j, S'''_j. Hence $dm_x(G) \geq t$. Since $S'_x = S \cup S'_1 \cup S'_2 \cup \cdots \cup S'_{s-r}$ is an x-detour monophonic set of G with $|S'_x| = t$, it follows that $dm_x(G) = t$.

Next, we prove that $f_{dm_x}(G) = r$ and $f^+_{dm_x}(G) = s$. Let T_x be any minimum x-forcing subset of S'_x. Then by Theorem 2.8, $|T_x| \leq t - (t - s) = s$. Since any subset of a minimum x-detour monophonic set with cardinality less than r contained in more than one minimum x-detour monophonic set of G, we have $f_{dm_x}(G) \geq r$. It is easily seen that $T'_x = \{v_{1,1}, v_{2,1}, \ldots, v_{(s-r),1}\}$ is a minimum x-forcing subset of minimum x-detour monophonic set $S'_x = S \cup S'_1 \cup S'_2 \cup \cdots \cup S'_{s-r}$ and $T''_x = \{v_{1,2}, v_{1,3}, v_{2,2}, v_{2,3}, \ldots, v_{(s-r),2}, v_{(s-r),3}\}$ is a minimum x-forcing subset of a minimum x-detour monophonic set $S''_x = S \cup S''_1 \cup S''_2 \cdots \cup S''_{s-r}$ of G. It follows that $f_{dm_x}(G) = r$ and $f^+_{dm_x}(G) = s$. $\square$

Problem 3.8. For any three positive integers r, s and t with $2 \leq r \leq s \leq t$ and $2r - s < 0$, does there exist a connected graph G with $f_{dm_x}(G) = r$, $f^+_{dm_x}(G) = s$ and $dm_x(G) = t$ for some vertex x in G?

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